Jumpstart Final Assessment

1. Prove or disprove: A set S is infinite if and only iff it has the same cardinality as a proper subset of itself.

2. Consider the sequence recursively defined by

$$\begin{cases} x_1 = 4, \\ x_{n+1} = \frac{5}{6-x_n}, \quad n \geq 1 \end{cases}$$

   Show that $(a_n)_{n\in\mathbb{N}}$ converges and determine its limit.

3. Let a sequence $(x_n)_{n\in\mathbb{N}}$ of positive real numbers be given. Prove or disprove: if $x_n \to x_\infty$ as $n \to \infty$, then the sequence defined by

$$y_n = \prod_{k=1}^{n} x_k^{\frac{1}{n}},\ n \in \mathbb{N},$$

   converges as well. If it does not, provide a counterexample; if it does, determine the limit.

4. State the definition of convergent series and of absolutely convergent series (of real numbers). Prove that that, if $\sum_{n\in\mathbb{N}} a_n$ is an absolutely convergent series, then the series $\sum_{n\in\mathbb{N}} a_n^2$ converges. Finally, give an example of a convergent series $\sum_{n\in\mathbb{N}} b_n$ such that $\sum_{n\in\mathbb{N}} b_n^2$ diverges.

5. Compute the following limit justifying your steps

$$lim_{n\to\infty} \int_0^\infty \frac{n\sin(\frac{x}{n})}{x(1+x^2)}\,dx.$$

6. Let $f : [0,1] \to \mathbb{R}$ be continuous and evaluate

$$lim_{n\to\infty} \int_0^1 x^n f(x)\,dx.$$

   Provide estimates that justify your answer.

7. Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence of functions in $\mathrm{C}\big([0,1]\big)$. Define

$$F_n(x) = \int_0^x f_n(\xi)\,d\xi,\ x \in [0,1],$$

   and show that $(F_n)_{n\in\mathbb{N}}$ has a uniformly convergent subsequence.

8. Let $E \subset \mathbb{R}$ be bounded and $f : E \to \mathbb{R}$ be uniformly continuous. Prove that f is bounded on E.

9. Suppose that X is an uncountable subset of the reals. Prove that there is a point of X that is a limit of a sequence of distinct points of X.