Qualifying Examination

Math Exam ID Number: ____________________

The exam consists of six problem. Solve as many as you can by giving complete arguments and providing explicit justification. Show all your work. Each problem is worth 5 points.

Problem	**Score**
1	
2	
3	
4	
5	
6	
Total	

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Problem 1

Let $(X, \mathcal{B}, \mu)$ be a measure space and $(f_n)_{n\in\mathbb{N}}$ a sequence in $L^1(\mu)$ that converges a.e. to $f \in L^1(\mu)$. Prove that $f_n \to f$ in $L^1(\mu)$ as $n \to \infty$ if and only if $\int |f_n| d\mu \to \int |f| d\mu$ as $n \to \infty$.

Problem 2

Suppose that $(X, \mathcal{B}, \mu)$ is a probability space and $\mathcal{C}$ is a σ-subalgebra of $\mathcal{B}$. Further suppose that $f \in L^1(X, \mathcal{B}, \mu)$.

(a) Prove that there is a unique $g \in L^1(X, \mathcal{C}, \mu)$ such that $\int_A f d\mu = \int_A g d\mu$ for all $A \in \mathcal{C}$.

(b) Suppose that $f = \chi_B$ for some $B \in \mathcal{B}$ and that $\mathcal{C} = \{\emptyset, C, C^c, X\}$ for some $C \in \mathcal{B}$ with $\mu(C) \in (0, 1)$. Write an explicit formula for the function g from part (a) in this case. Make sure to justify your answer.

Problem 3

Let $f \in \mathrm{L}^1_{loc}(\mathbb{R}^n)$, $p \in (0,1)$, and assume that

$$\Big| \int fg\,dx \Big| \le \Big(\int |g|^p \Big)^{1/p} \quad \forall g \in \mathrm{C}_c(\mathbb{R}^n).$$

Conclude that $f = 0$ a.e.
[Hint: Show that the inequality extends to characteristic functions of balls.]

Problem 4

Let ν_1, ν_2 be finite signed measures on a measurable space $(X, \mathcal{M})$ and show that

$$\big|\nu_1 + \nu_2\big|(E) \leq |\nu_1|(E) + |\nu_2|(E) \text{ for } E \in \mathcal{M}.$$

Also prove the inequality can be strict.

Problem 5

Let $f \in \mathrm{L}^4(\mathbb{R})$ and show that

$$\lim_{c\to 1} \int |f(cx) - f(x)|^4 \, dx = 0$$

Problem 6

Let $E_1, \ldots, E_{10} \subset [0,1]$ be the Lebesgue measurable sets. Suppose that each $x \in [0,1]$ belongs to at least 5 of these sets (which 5 can vary with x). Prove that $\lambda(E_m) \geq 1/2$ for some $m \in \{1, \ldots, 10\}$, where λ is the Lebesgue measure on the real line.