Print Your Name: ______________ ________________
last first

Print Your I.D. Number: __________ ______________

Real Analysis Qualifying Examination, Wednesday, June 21, 2023

1:00 PM – 3:30 PM, Room RH 306

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1. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f : X \to \mathbb{R}$ be a measurable function.

(a) Prove that, for any $r > 1$, one has

$$\int |f|^r d\mu = r \int_0^\infty t^{r-1} \mu(\{x \in X \ : \ |f(x)| > t\}) dt.$$

(b) Suppose that $1 \leq p < r < q < \infty$ and there is $C < \infty$ such that

$$\mu(\{x \mid |f(x) > \lambda\}) \leq \frac{C}{\lambda^p + \lambda^q}$$

for every $\lambda > 0$. Prove that $f \in L^r(\mu)$.

2. Recall that the Cantor-Lebesgue function is the function $C : [0,1] \to [0,1]$ first defined on the Cantor set by setting $C(x) = \sum_{j=0}^{\infty} b_j 2^{-j}$, where $b_j = \frac{a_j}{2}$ and $x = \sum_{j=0}^{\infty} a_j 3^{-j}$, and then continuously extended to all of $[0,1]$ by setting it to be constant on the intervals deleted in the formation of the Cantor set. Let μ_C be the Borel measure on $\mathbb{R}$ defined by

$$\mu_C([a,\, b)) = C(b) - C(a)$$

for all $a < b$. Prove that μ_C is not absolutely continuous with respect to Lebesgue measure.

3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is integrable and $g : \mathbb{R} \to \mathbb{R}$ is bounded and measurable. Prove that

$$\lim_{t \to 0} \int_{-\infty}^{\infty} g(x) \cdot [f(x) - f(x+t)] dx = 0.$$

4. Suppose that $f \in L^1([0,1])$ is nonnegative. Prove that

$$\lim_{n\to\infty} \int_0^1 \sqrt[n]{f(x)}\, dx = m(\{x \in [0,1] \ : \ f(x) > 0\}).$$

5. Let $E \subseteq \mathbb{R}$ be a Lebesgue measurable set such that $E + q \subseteq E$ for each $q \in \mathbb{Q}$. Prove that $m(E) = 0$ or $m(E^c) = 0$, where m denotes Lebesgue measure.

6. Fix a measure space $(X, \mathcal{M}, \mu)$, $1 \leq p < \infty$, and functions $f_n, f \in L^p(X)$ such that $f_n \to f$ almost everywhere. Prove that the following two conditions are equivalent:

(a) $f_n \to f$ in $L^p(X)$.

(b) $\|f_n\|_p \to \|f\|_p$.