Complex Analysis Qualifying Exam

September 15, 2022

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Problem 1: Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function such that $|\mathrm{Re}(f(z))| < 1$ for all $z \in \mathbb{D}$. Show that $|f'(0)| \leq 2$.

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Problem 2: Let f be an entire function. Show that the following series converges uniformly on compact subsets of $\mathbb{C}$:

$$\sum_{n=1}^{\infty} \frac{f^{(n)}(z)}{n^n}.$$

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Problem 3: Let f be an entire function. Suppose the family

$$\mathcal{F} = \{f_n;\ f_n(z) = f(nz)\},$$

is a normal family on the annulus $\{1 < |z| < 2\}$. Show that f is a constant.

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Problem 4: Let p, q be polynomials on $\mathbb{C}$ and assume that

$$p(0) = 0, \quad p'(0) \neq 0,$$

and $|p(z)| > 0$, for all $0 < |z| \leq 1$.

(i) Show that there exists $\delta > 0$ such that for all $\varepsilon \in \mathbb{C}$, $|\varepsilon| < \delta$, the polynomial $z \mapsto p(z) + \varepsilon q(z)$ has a unique root $z(\varepsilon) \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, which is simple.

(ii) Prove that the function $\varepsilon \mapsto z(\varepsilon)$ is holomorphic on $\{\varepsilon \in \mathbb{C} : |\varepsilon| < \delta\}$.

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Problem 5: Let $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Determine all holomorphic functions $f : \mathbb{C}_+ \to \mathbb{C}$ that satisfy $f(i\sqrt{n}) = n$ and $|f^{(n)}(i)| \leq 3$ for $n = 1, 2, \ldots$.

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Problem 6: Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let f be holomorphic in a neighborhood of $\overline{\mathbb{D}}$, which satisfies

$$|f(0)| + |f'(0)| < \inf\{|f(z)| : |z| = 1\}.$$

Show that f has at least two zeros (counting multiplicity) in $\mathbb{D}$.

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Problem 7: Let $\Omega \subset \mathbb{C}$ be open bounded simply connected and let $f : \Omega \to \Omega$ be holomorphic such that $f(0) = 0$, $|f'(0)| < 1$. Let

$$f^{(n)} = f \circ f \circ \cdots \circ f$$

be the n-fold composition of f with itself, $n = 1, 2, \ldots$ Show that $f^{(n)} \to 0$ uniformly on compact subsets of Ω, as $n \to \infty$.

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Problem 8: Suppose f and g are entire functions with no common zeros. Show that there exist entire functions F and G such that

$$fF + gG = 1.$$