Applied Mathematics Qualifying Exam

September 21, 2023

Time limit: 2.5 hours

Instructions: This exam has three parts A, B, and C, each of which contains three problems. Choose TWO problems from each of Parts A and C, and in Part B, you MUST do Problem 1 and then choose ONE of problems 2 and 3, for a total of SIX problems.

Part A

Choose any TWO of the following problems.

1. Consider the planar ODE

$$\begin{aligned} u' &= v \\ v' &= 2v - u(1-u)^2. \end{aligned} \tag{1}$$

   (a) Find the linearization of (1) at the fixed points $p_0 = (0,0)$ and $p_1 = (1,0)$.

   (b) Show that the triangular region $\mathcal{T}$ bounded by the lines $v = 0$, $u = 1$, and $v = u$, is negatively invariant under the flow of (1).

   (c) Show that (1) admits a heteroclinic orbit between the fixed points p_0 and p_1 which approaches p_1 along a center manifold.

2. Consider the boundary value problem

$$\begin{aligned} \varepsilon y'' + a(x)y' + b(x)y = 0, \qquad 0 < x < 1 \\ y(0) = y_0, \quad y(1) = y_1, \end{aligned}$$

   where $a(x) > 0$ for $x \in [0,1]$. Using WKB theory, show that a leading order asymptotic expansion for the solution is given by

$$y(x) \sim C_1 e^{-\int_0^x (b(s)/a(s))\mathrm{d}s} + \frac{C_2}{a(x)} e^{\int_0^x (b(s)/a(s))\mathrm{d}s - \frac{1}{\varepsilon}\int_0^x a(s)\mathrm{d}s}$$

   where

$$C_1 = y_1 e^{\int_0^1 (b(s)/a(s))\mathrm{d}s}, \qquad \text{and} \qquad C_2 = a(0)(y_0 - C_1).$$

3. Consider the initial value problem

$$\dot{u} = f(u), \qquad u(0) = u_0. \tag{2}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz. That is, there exists $K, \delta > 0$ such that $|f(u) - f(v)| \leq K|u - v|$ for all $u, v \in B_\delta(u_0) = \{u \in \mathbb{R}^n : |u - u_0| < \delta\}$.

(a) Consider the map T defined by

$$(Tu)(t) = u_0 + \int_0^t f(u(s))\mathrm{d}s.$$

and take $\varepsilon < \min\left\{\frac{1}{K}, \frac{\delta}{M}\right\}$ where $M = \sup_{u \in B_\delta(u_0)} |f(u)|$. Show that T is a contraction on the space $\mathcal{B} = \{u \in C^0([-\varepsilon, \varepsilon], \mathbb{R}^n) : \sup_{t \in [-\varepsilon, \varepsilon]} |u(t) - u_0| < \delta\}$, where $C^0([-\varepsilon, \varepsilon], \mathbb{R}^n)$ denotes the space of continuous functions with the supremum norm $\|u\| = \sup_{t \in [-\varepsilon, \varepsilon]} |u(t)|$. What does this imply about solutions to the initial value problem (2)? (You may use the contraction mapping theorem without proof.)

(b) Now suppose $f \in C^1(\mathbb{R}^n)$ and that the maximal interval of existence of the solution to the initial value problem (2) is $(-\infty, \beta)$ where $\beta < \infty$. Show that $|u(t)|$ is unbounded as $t \to \beta$.

Part B

You must complete problem 1, and then choose ONE of problems 2 or 3.

1. (**Mandatory**) Numerical ODE problem.

Consider a linear multistep scheme of the form

$$\begin{aligned} w_{n+1} &= a_1 w_n + a_2 w_{n-1} + h\left(b_0 f(t_{n+1}, w_{n+1}) + b_1 f(t_n, w_n)\right), \quad n \geq 1, \\ w_1 &= y_1, \\ w_0 &= y_0, \end{aligned}$$

for solving the ODE: $y' = f(t, y)$ for $0 < t \leq T$ and $y(0) = y_0$. Here, $h = T/N$ is the time step, N is the total number of time steps, $t_n = nh$ and $\mathbf{w}_n$ is the numerical approximation to $\mathbf{y}(t)$ at $t = t_n$.

(a) Find the equation for the error $\mathbf{e}_n = \mathbf{y}_n - \mathbf{w}_n$, e.g., that describes how the error propagates in time, and describe the meaning of each of the terms in the equation.

(b) What is the highest order of accuracy this method can attain? Determine the coefficients a_1, a_2, b_0, b_1 that make the scheme reach this order of accuracy, assuming y_1 is a sufficiently accurate approximation of $y(h)$. **Hint:** The scheme is pth order accurate if the test functions $y(t) = 1, t, t^2, \ldots t^p$, and $f(t, y(t)) = 0, 1, 2t, \ldots, pt^{p-1}$ satisfy the scheme exactly with $h = 1$.

(c) How do you determine if the method is stable? Hint: Don't forget that this is a multistep method.

(d) Define A-stability for a numerical scheme for first order ODEs. Is the optimally accurate scheme you found A-stable?

2. Least squares problem.

   (a) Let A be a real $m \times n$ matrix. State the singular value decomposition (SVD) of A. Briefly present an algorithm to solve the overdetermined system $(m > n)$ using the SVD.

   (b) What are the advantages and disadvantages of using the SVD to solve overdetermined systems?

   (c) Perform the SVD on the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$.

   (d) Compute the pseudo-inverse of A (the Moore-Penrose pseudo-inverse). Leave in factored form.

   (e) Find the minimal length, least squares solution of the problem: $A\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

3. Eigenvalue problem.

   (a) Let A be a $n \times n$ matrix. Prove Gerschgorin's theorem, which states: For $i = 1, \ldots, n$ let $R_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$. Every eigenvalue of A falls within at least one of the closed discs in the complex plane with center at a_{ii} and radius R_i. **Hint:** Let

$$A\mathbf{x} = \lambda \mathbf{x} \tag{3}$$

   and assume that the largest component of $\mathbf{x}$ in absolute value is x_k. Consider the kth equation of Eq. (3).

   (b) Consider the matrix $A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$. Show that A is positive definite.
   Hint: Use Gerschgorin's theorem to show that A is positive semi-definite. Then consider the equation where $A\mathbf{x} = \mathbf{0}$, where the first component of $\mathbf{x}$ equals 1.

   (c) Let $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. Find the eigenvalues of A and verify that Gerschgorin's theorem holds.

Part C

Choose any TWO of the following problems.

1. Let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(v) = v[1 + \min\{v, 0\}]$.

   (a) Show that g is convex, continuously differentiable, and satisfies

   $$g(v) \geqslant \max\{v, |v| - 1\}.$$

   (b) Deduce that any arc x admissible for the problem

   $$\min J(x) = \int_0^1 \left(x^2 + g\left(x'\right)\right) \mathrm{d}t :$$

   subject to $x \in \mathrm{AC}[0, 1], x(0) = 0, x(1) = 1$ satisfies $J(x) > 1$.

   (c) Show that the functions

   $$x_i(t) = \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant 1 - 1/i \\ i[t - 1 + 1/i] & \text{if } 1 - 1/i < t \leqslant 1 \end{cases}$$

   satisfy $\lim_{i\to\infty} J\left(x_i\right) \to 1$.

   (d) Conclude that the problem defined in (b) admits no solution.

2. Assume $f \in L^2(\Omega)$.

   (a) Prove the dual variational principle that

   $$\min_{v\in H_0^1(\Omega)} \int_\Omega (\frac{1}{2}|\nabla v|^2 - fv)\, \mathrm{d}x = \max_{\substack{\sigma\in L^2(\Omega;\mathbb{R}^n) \\ \operatorname{div}\sigma=f}} -\frac{1}{2}\int_U |\sigma|^2 \mathrm{d}x.$$

   (b) Write out Euler-Lagrange equations for both formulations.

3. For each of the given Lagrangian, find the Hamiltonian and solve the Hamiltonian system.

   (a) $L(t, x, v) = (v + kx)^2, k \neq 0$.

   (b) $L(t, x, v) = e^{-x}\sqrt{1 + v^2}$.