Analysis Comprehensive Exam

June 26, 2023

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SCORES:

1. __________ /10

2. __________ /10

3. __________ /10

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9. __________ /10

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Problem 1: Let A be an infinite closed subset of $\mathbb{R}^n$. Show that there exists a countable set whose closure is A.

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Problem 2: Consider the sequence recursively defined by

$$\begin{cases} x_{n+1} = \frac{x_n^2+1}{2}, \quad n \geq 1, \\ x_1 = 0. \end{cases}$$

Show that $\{x_n\}_{n\in\mathbb{N}}$ converges and determine its limit.

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Problem 3: Show that the sum

$$\sum_{n\in\mathbb{N}} \frac{n^2}{n^4 + x^4}$$

converges uniformly on $[-2, 2]$ to a continuous function denoted by f. Give a formula for $f'(x)$ and prove that your formula is correct.

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Problem 4: Let $f, g : [0,1] \to [0,\infty)$ be continuous functions satisfying

$$\sup_{0\leq x\leq 1} f(x) = \sup_{0\leq x\leq 1} g(x)$$

Prove that there exists $x_0 \in [0,1]$ such that $f(x_0) = g(x_0)$.

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Problem 5: Let $\{b_n\}$ be a monotonic increasing sequence of positive numbers and $\lim_{n\to\infty} b_n = +\infty$. Show that, if $\sum_{n=1}^{\infty} a_n$ converges, then

$$\lim_{n\to\infty} \frac{a_1 b_1 + \cdots + a_n b_n}{b_n} = 0.$$

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Problem 6: Let f be a real-valued continuous function on $[0,1]$ that is differentiable on $(0,1)$. Assume

$$\sup_{0<x<1} |f'(x)| = M < \infty.$$

Show that for all $n \in \mathbb{N}$:

$$\left| \frac{1}{n} \sum_{j=0}^{n-1} f(j/n) - \int_0^1 f(x)\,dx \right| \leq \frac{M}{n}.$$

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Problem 7: Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x = 0$ and satisfies

$$\lim_{x \to 0} \frac{f(2x) - f(x)}{x} = m.$$

Show that $f'(0) = m$.

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Problem 8: Let $f, g : \mathbb{R} \to \mathbb{R}$ be smooth functions with $f(0) = 0$ and $f'(0) \neq 0$. Consider the equation

$$f(x) = tg(x), \qquad t \in \mathbb{R}.$$

Show that there exists $\delta > 0$ such that on the interval $|t| < \delta$, there is a unique smooth solution $x(t)$ to the above equation with $x(0) = 0$. Then calculate $x'(0)$ and $x''(0)$.

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Problem 9: Compute the volume of the balls

$$B_n(r) = \{x \in \mathbb{R}^n : x_1^2 + x_2^2 + \cdots + x_n^2 \leq r^2\},$$

for $n = 3$ and $n = 4$.