Degenerate Diffusion in Heterogeneous Media

Guillermo Reyes Souto

ABSTRACT

In this talk, I will present some recent results, obtained in collaboration with J.L. Vázquez (Madrid) and S. Kamin (Tel-Aviv), on the long-time behavior of non-negative solutions to the Cauchy problem for the Porous Medium Equation in the presence of variable density, *i.e.*

$$(\mathbf{P}) \qquad \begin{cases} \rho(x)\,\partial_t u = \Delta u^m & \text{in } \ Q := \mathbb{R}^n \times \mathbb{R}_+ \\ u(x,\,0) = u_0(x) & \text{in } \ \mathbb{R}^n \end{cases}$$

We assume $m > 1$ and $n \geq 3$.

Both the basic theory and the long-time behavior of solutions in the homogeneous case $\rho = \text{const.}$ are well understood by now.

Here, $\rho(x)$ is positive, bounded and has a power-like decay at infinity,

$$\rho(x) \sim |x|^{-\gamma} \qquad \text{as } \ |x| \to \infty$$

for some $\gamma > 0$. The data u_0 are assumed to be non-negative and such that

$$\int_{\mathbb{R}^n} \rho(x) u_0(x)\, dx < \infty$$

(finite-energy solutions).

Interestingly, the behavior of solutions depends on whether $0 < \gamma < 2$ or $\gamma > 2$. In the first case, the asymptotic behavior is described in terms of a one-parameter family of source-type, self-similar solutions of a related singular problem (so called Barenblatt-type solutions, $U_E(x,t)$), thus representing a natural extension of the corresponding result for the standard PME.

For $\gamma > 2$, however, solutions to $(\mathbf{P})$ have a *universal* long-time behavior in *separate variables*, typical of initial-boundary problems on bounded domains. Thus, the presence of a rapidly decreasing density has the effect of "compactifying" the domain.

If $\rho(x)$ has an intermediate decay, $\rho(x) \sim |x|^{-\gamma}$ with $2 < \gamma < \gamma_2 := N - (N-2)/m$, solutions still enjoy the finite propagation property (as in the case of lower γ). In this range a more precise description may be given at the diffusive scale in terms of the Barenblatt-type solutions $U_E(x,t)$ of the related singular equation $|x|^{-\gamma} u_t = \Delta u^m$. Thus in this range we have *two* different space-time scales in which the behavior of solutions is non-trivial. The corresponding results complement each other and agree in the intermediate region where both apply, thus providing an example of matched asymptotics.