Applied Mathematics Qualifying Exam

June 23, 2023

Time limit: 2.5 hours

Instructions: This exam has three parts A, B, and C, each of which contains three problems. Choose TWO problems from each of Parts A and C, and in Part B, you MUST do Problem 1 and then choose ONE of problems 2 and 3, for a total of SIX problems.

Part A

Choose any TWO of the following problems.

1. Consider the following boundary value problem, where $\varepsilon > 0$ is a small parameter,

$$\begin{aligned} &\varepsilon y'' + \varepsilon(2-x)y' - y = -x^2, \qquad 0 < x < 1 \\ &y(0) = 1, \qquad y(1) = 0 \end{aligned}$$

   Find a leading order asymptotic solution to the equation using boundary layer theory.

2. In the equation below, determine whether the fixed point $(x, y) = (0, 0)$ is locally asymptotically stable for each value of the parameter p. Clearly state any theorems you use.

$$\begin{aligned} \dot{x} &= px + x^3 + y\sin x \\ \dot{y} &= -y - 2x^2. \end{aligned}$$

   What kind of bifurcation occurs at $p = 0$?

3. By converting to polar coordinates, show that the system

$$\begin{aligned} \dot{x} &= x + y - x(2x^2 + 3y^2) \\ \dot{y} &= -x + y - y(2x^2 + 3y^2) \end{aligned}$$

   has at least one periodic orbit. Clearly state any theorems you use. Do you expect the periodic orbit to be locally asymptotically stable? Why or why not?

Part B

You must complete problem 1, and then choose ONE of problems 2 or 3.

1. (**Mandatory**) Numerical ODE problem.

   Consider a system of ODEs

$$\begin{aligned} \mathbf{y}' &= \mathbf{f}(t, \mathbf{y}),\ 0 \le t \le T \\ \mathbf{y}(0) &= \mathbf{y}_0 \end{aligned} \tag{1}$$

   and the midpoint method:

$$\mathbf{k}_1 = h\mathbf{f}(t_n, \mathbf{w}_n), \tag{2}$$

$$\mathbf{w}_{n+1} = \mathbf{w}_n + h\mathbf{f}(t_n + \frac{h}{2}, \mathbf{w}_n + \frac{1}{2}\mathbf{k}_1),\ n \ge 0, \tag{3}$$

$$\mathbf{w}_0 = \mathbf{y}_0, \tag{4}$$

   where $h = T/N$ is the time step, N is the total number of time steps, $t_n = nh$ and $\mathbf{w}_n$ is the numerical approximation to $\mathbf{y}(t)$ at $t = t_n$.

   (a) Find the equation for the error $\mathbf{e}_n = \mathbf{y}_n - \mathbf{w}_n$, e.g., that describes how the error propagates in time, and describe the meaning of each of the terms in the equation.

   (b) Determine the order of accuracy of the method.

   (c) How do you determine if the method is stable?

   (d) Define and determine the region of absolute stability for this method.

2. Least squares problem.

   (a) Let A be a real $m \times n$ matrix. Describe the singular value decomposition (SVD) of A. Provide an explanation of the rank of A and how the SVD relates to the four fundamental subspaces $\mathcal{R}(A)$, the range of A, $\mathcal{R}(A^T)$, the range of A^T, $\mathcal{N}(A)$, the nullspace of A and $\mathcal{N}(A^T)$, the nullspace of A^T.

   (b) Perform the SVD on the matrix $A = \begin{pmatrix} 2 & 3 \\ 2 & -3 \\ 1 & 0 \end{pmatrix}$

   (c) Compute the pseudo-inverse of A (the Moore-Penrose pseudo-inverse). Leave in factored form.

   (d) Find the minimal length, least squares solution of the problem: $A\mathbf{x} = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$.

3. Eigenvalue problem.

   (a) Let λ be an eigenvalue of a $n \times n$ matrix A. Show that $p(\lambda)$ is an eigenvalue of $p(A)$ for any polynomial $p(x) = \sum_{j=1}^{n} c_j x^j$.

   (b) Let A be a $n \times n$ symmetric matrix whose components satisfy $a_{1i} \neq 0$, $\sum_{j=1}^{n} a_{ij} = 0$, $a_{ii} = \sum_{j \neq i} |a_{ij}|$ for $i = 1, \ldots, n$. Show all the eigenvalues of A are non-negative and determine the dimension of the eigenspace corresponding to the smallest eigenvalue of A. Hint: Use Gershgorin's theorem.

Part C

Choose any TWO of the following problems.

1. Find the weak minimum of the following functional.

   (a) $I(x) = \int_0^1 t\dot{x} + \dot{x}^2 \,\mathrm{d}t \quad x \in C_0^1[0,1]$.

   (b) $I(x) = \int_0^b x^2 + \dot{x}^2 \,\mathrm{d}t, \quad \mathcal{M} = \{x \in C^1[0,b] \mid x(0) = 0, x(b) = B\}$.

   Please justify the extreme curve is a local weak minimum.

2. Let $L(t, x, v) = \frac{1}{2}v^2 - vx$ be the Lagrangian.

   (a) Find Hamiltonian and solve the Hamilton system.

   (b) Write out Hamilton-Jacobi equation and solve it.

3. Assume $F \in C^2$ and $|F''| \le \lambda_1$, where λ_1 is the minimal eigenvalue of $-\Delta$ operator with homogenous Dirichlet boundary condition on Ω. Consider

$$\min_{u \in H_0^1(\Omega)} I(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + F(u) \right) \mathrm{d}x.$$

   Assume $u_0 \in H_0^1(\Omega)$ solves the Euler-Lagrange equation. Prove it is minimum.