ALGEBRA QUALIFYING EXAM
SEPTEMBER 2023

Math Exam ID#: ____________________

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total		100

Instructions: All problems are worth 10 points. Unless otherwise specified, all rings are commutative, have unity, are not the zero ring, and all ring homomorphisms map unity to unity. Write your proofs clearly using complete sentences. Your proofs will be graded based on clarity as well as correctness; a correct answer will not receive full credit if the reasoning is difficult to follow. Good luck.

1. For each of the following group actions of the symmetric group S_4, describe the indicated stabilizer as a subgroup of S_4 (for example, by giving generators for the subgroup). No explanation is necessary.
   a. The natural action of S_4 on $\{1, 2, 3, 4\}$. Identify the stabilizer of 2.
   b. The action of S_4 on itself by left-multiplication: $(g, h) \mapsto gh$.
   Identify the stabilizer of (23).
   c. The action of S_4 on itself by conjugation: $(g, h) \mapsto ghg^{-1}$.
   Identify the stabilizer of (23).
2. Let G be a group whose group of automorphisms is cyclic. Prove that G is abelian.
   Hint: You may use the following without proof. If G is a group such that $G/Z(G)$ is cyclic, where $Z(G)$ denotes the center of G, then G is abelian.
3. Let R be a non-zero finite commutative ring with unity. Prove that every prime ideal in R is maximal.
4. For this question, be sure to recall our assumption that any ring homomorphism maps unity to unity.
   a. Describe all ring homomorphisms from $\mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$. Justify your answer.
   b. Describe all ring homomorphisms from $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$. Justify your answer.
5. Let R be a ring and let M be an R-module. Prove that M is a finitely generated R-module if and only if M is isomorphic to a quotient of a free R-module of finite rank.
6. Is there a 14×14 matrix A with integer entries and order 49; that is, $A^{49} = I$ but $A^k \neq I$ for $0 < k < 49$? Justify your answer.
7. Prove that $x^5 - 2$ is irreducible over $\mathbb{Q}(\zeta_5)$.
8. What are the cardinalities of the following subsets of $\mathbb{F}_{3^6}$? Here $\mathbb{F}_{3^6}$ denotes a field with 3^6 elements. Justify your answers. (Note: $3^6 = 729$.)
   a. $\{x^{5^4} : x \in \mathbb{F}_{3^6}\}$
   b. $\{x \in \mathbb{F}_{3^6} : x^{3^4} = x\}$
9. Assume K/F is a Galois extension and $\text{Gal}(K/F) \cong Q_8$, the quaternion group of order 8. Does there exist a degree 4 polynomial $f(x) \in F[x]$ such that K is the splitting field over F of $f(x)$? Justify your answer.
10. Let p be a prime, and let $K/\mathbb{Q}$ denote a splitting field of a degree p polynomial $f(x) \in \mathbb{Q}[x]$ which is irreducible over $\mathbb{Q}$. Let α, β denote distinct roots of $f(x)$, and assume $\mathbb{Q}(\alpha) = \mathbb{Q}(\beta)$. Prove that $\text{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/p\mathbb{Z}$.