Print Your Examination number: ______________

Complex Qualifying Examination

1:00pm–3:30pm, June 22, 2023

Table of your scores

Problem 1 ________/ 10

Problem 2 ________/ 10

Problem 3 ________/ 10

Problem 4 ________/ 10

Problem 5 ________/ 10

Problem 6 ________/ 10

Problem 7 ________/ 10

Problem 8 ________/ 10

Problem 9 ________/ 10

Total ________/ 90

Notation. Let $D(a,r)$ denote the disc in $\mathbf{C}$ centered at a with radius r. Let $\mathbf{H} = \{z \in \mathbf{C} : \mathrm{Im} z > 0\}$ be the upper half plane.

1. Evaluate the following integral

$$\frac{1}{2\pi i} \int_{|z|=4} \frac{\cos\left(e^z\right)}{\sin^2(z)}\,\mathrm{dz}.$$

2. Let f be a non-constant analytic function on the closed unit disk $\overline{D(0,1)}$. Suppose that $|f(z)| = 1$ if $|z| = 1$. Prove that $f(D(0,1)) = D(0,1)$.

3. Let

$$J(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{[(n)!]^2} \left(\frac{z}{2}\right)^{2n}.$$

Prove that $J(z)$ is entire and satisfies that

$$zJ''(z) + J'(z) + zJ(z) = 0.$$

4. Prove that for any $a \in \mathbf{C}$ and any integer $n \geq 2$, the polynomial $2022 + az + 2023z^n$ has at least one root in the unit disk $D(0, 1)$.

5. Let f be a holomorphic function in the unit disk $D(0, 1)$ that is injective and satisfies $f(0) = 0$. Prove that there exists a holomorphic function g in $D(0, 1)$ such that $(g(z))^{2023} = f(z^{2023})$ for all $z \in D(0, 1)$.

6. Let $f_1 : D(0,1) \to D(0,1)$ be a non-constant holomorphic function and define inductively

$$f_{n+1}(z) = \frac{1}{1+f_n(z)} : D(0,1) \to \mathbf{C}, \text{ for } \ n \in \mathbf{N}.$$

a) Show that for each $n \in \mathbf{N}$, the function f_n is a holomorphic function in the unit disc $D(0,1)$.

b) Does the sequence of holomorphic functions $\{f_n\}_{n\in\mathbf{N}}$ form a normal family in $D(0,1)$? Explain your answer.

7. Let $u(z) = u(x, y) = x^3 - 3xy^2 - 6xy$. Find all entire functions $f(z)$ such that $\text{Re} f(z) = u(z)$.

8. True or False. There is a sequence of holomorphic functions $\{f_n\}_{n=1}^{\infty}$ on the unit disc $D(0,1)$ such that $f_n(z) \to \cos(\overline{z}^2)$ as $n \to \infty$ uniformly on the circle $|z| = \frac{1}{2}$.

9. Let $\mathcal{F}$ be the family of all holomorphic functions

$$f : \mathbf{H} \to \Omega = \{z \in \mathbf{C} : |z| > 1\} \text{ such that } f(i) = 2.$$

Give the best estimate for $|f(3i)|$ for $f \in \mathcal{F}$.