Print Your Examination number: ______________

Complex Qualifying Examination

1:00pm–3:30pm, September 20, 2023 at RH 306

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Notation. Let $D(a, r)$ denote the disc in the complex plane $\mathbf{C}$ centered at a with radius r and $\partial D(a, r) = \{z \in \mathbf{C} : |\mathbf{z} - \mathbf{a}| = \mathbf{r}\}$. Let $\mathbf{R}$ denote the set of all real numbers. Let $\mathbf{H} = \{z \in \mathbf{C} : \mathrm{Im} z > 0\}$ be the upper half plane.

Problem 1: Find the line integral:

$$\int_\gamma z \sin z \, dz$$

where γ the curve from -1 to 1 taken along a semicircle.

Problem 2: A domain Ω in $\mathbf{C}$ is said to be holomorphic simply connected if for any holomorphic function f on Ω and any simple closed piecewise C^1 curve γ in Ω, one has $\int_\gamma f(z)dz = 0$.

(a) Prove that

$$\Omega = \{z = x + iy \in \mathbf{C} : y > x\}$$

is holomorphic simply connected.

(b) Prove that $\Omega = D(0,1) \setminus \{0\}$ is not holomorphic simply connected.

Problem 3:

a) Prove that the series $\sum_{n=0}^{\infty} e^{n(1+i)z}$ converges to a holomorphic function in a neighborhood of the point $z_0 = i$.

b) If $f(z) = \sum_{n=0}^{\infty} e^{n(1+i)z}$ is represented as a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - i)^n,$$

what is its radius of convergence?

Problem 4: Find an explicit conformal mapping of the domain

$$U = \{z \in \mathbf{C} \ : \ |z - i| < \sqrt{2} \ \text{ and } \ |z + i| < \sqrt{2}\}$$

onto the unit disc.

Problem 5: If $f : \mathbf{C} \to \mathbf{C}$ is an entire function, one can define a sequence of functions $\left\{f^{(n)}\right\}$ by

$$f^{(1)} = f,\ f^{(2)} = f \circ f,\ f^{(3)} = f \circ f \circ f, \ldots,\ f^{(n)} = f^{(n-1)} \circ f, \ldots$$

Prove or disprove: For any entire function $f : \mathbf{C} \to \mathbf{C}$, the sequence $\left\{f^{(n)}|_{D(0,1)}\right\}$ of restrictions of $f^{(n)}$ to the unit disc form a normal family.

Problem 6:

a) Suppose that $u : \mathbf{C} \to \mathbf{R}$ is a harmonic function such that $u|_{\mathbf{R}} = 0$. Does it imply that $u \equiv 0$?

b) Suppose that $u : \mathbf{C} \to \mathbf{R}$ is a harmonic function such that $u|_{\partial D(0,1)} = 0$. Does it imply that $u \equiv 0$?

Problem 7: Let f be entire holomorphic such that

$$|f(z)|^2 = p_n(x, y), \quad z = x + iy,$$

where $p_n(x, y)$ is a polynomial of x and y of degree n. Prove that f is a polynomial of z.

Problem 8: Let f be holomorphic in $D = D(0,1) \setminus \{0\}$ such that

$$\int_0^{2\pi} |f(re^{i\theta})| d\theta \leq 1, \text{ for all } 0 < r < 1.$$

Prove that $z = 0$ is a removable singularity.