Algebra Qualifying Examination[1]

September 2000

The 8 exam problems: Do as many problems as you can. We prefer complete solutions of a few problems to many partial solutions. 60 total points is sufficient to pass at the Master's Level. 75 total points is sufficient to pass at the PhD level. There are 2 more problems at the end. If you do them you will get additional credit. Do each problem on a separate page. Write your name on each page. If you don't understand some terminology, please ask. The notation throughout a given problem remains constant. Show all the details and quote the theorems you use properly. In front of every section of each problem we show how many points it is worth. You have 150 minutes. GOOD LUCK!.

Notation: S_n is the symmetric group on n integers. The invertible $n \times n$ matrices over a field K is $\mathrm{GL}_n(K)$. For R a ring, R^* is the units of R.

1 Homomorphisms of Groups

The group S_3 acts on $\{1, 2, 3\}$.

1.a (3) How does this give a homomorphism of S_3 into $GL_3(\mathbb{C})$ (acting on $V = \mathbb{C}^3$) with trivial kernel?

1.b (2) We will say that $V = \mathbb{C}^3$ is an irreducible vector space under the action of S_3 iff the only vector in V fixed under the action of S_3 is the zero vector. Is $V = \mathbb{C}^3$ an irreducible vector space under the action of S_3?

1.c (5) Find V_1 and V_2 – two different vector subspaces of V preserved by the image of S_3 and such that V is a direct sum of V_1 and V_2.

1.d (5) Find the matrices by which S_3 acts on V_1 and V_2.

2 The Quaternion Group

Suppose $G = Q_8$ is the quaternion group. Here are the properties of G: it is non-abelian; it is of order 8, and it has exactly one element of order 2.

2.a (2) Explain why every subgroup of G is normal.

2.b (3) Count the number of conjugacy classes in G.

2.c (5) Let N be a normal subgroup in K and K a normal subgroup in L. TRUE OR FALSE: N is a normal subgroup of L? Explain.

[1] Transcribed by Joshua Hill, 2014-04-01. Providence unknown. Likely a UCI qualifying exam.

3 Irreducibility of a polynomial

Let p be a prime number.

3.a (5) Show that the polynomial

$$x^4 + 15x^3 + 20x^2 + 10x + 45$$

is irreducible over $\mathbb{Q}$.

3.b (7) Show that the polynomial

$$x^{p-1} + x^{p-2} + \cdots + 1$$

is irreducible over $\mathbb{Q}$. (Hint: Relate it to the polynomial $x^p - 1$.)

4 The group $\mathbb{F}_p^*$

Let $\mathbb{F}_p$ be a field of p elements.

4.a (7) Prove that $\mathbb{F}_p^*$ is a cyclic group.

4.b (5) Using a) prove that $(p-1)! \equiv -1 \pmod{p}$.

5 The Group SO$(3, \mathbb{R})$

Show that the group $SO(3, \mathbb{R})$ is a simple group. Use without a proof that there exists a surjective homomorphism $\rho : SU(2) \to SO(3, \mathbb{R})$ such that $\ker(\rho) = \mathbb{Z}_2$.

5.a (3) Describe the groups $SO(3, \mathbb{R})$ and $SU(2)$.

5.b (4) Describe all conjugacy classes of $SU(2)$.

5.c (3) Prove that if H is a normal subgroup of $SU(2)$ it is a union of conjugacy classes of $SU(2)$.

5.d (3) Let H be a normal subgroup of $SU(2)$ and $h \in H$. Let a be an arbitrary element of $SU(2)$. Compute the commutator c of a and h. Show that c belongs to H.

5.e (4) Using part d) of the problem show that $SU(2)$ has only one nontrivial normal subgroup.

6 Abelian groups

6.a (6) Determine the direct sum structure of the abelian group A generated by $\{x, y, z\}$ with the following three relations

$$7x + 5y + 2z = 0, 10x + 8y + 2z = 0, 13x + 11y + 2z = 0.$$

6.b (6) Describe all abelian groups of order 72.

7 Burnside's formula

Let G be a finite group acting on a finite set S. For each element $g \in G$, let $S^g = \{s \in S \mid g(s) = s\}$ be the subset of elements of S fixed by g. For $s \in S$, let $G_s = \{g \in G \mid g(s) = s\}$ be the stabilizer of s.

7.a (6) Prove the formula $\sum_{s \in S} |G_s| = \sum_{g \in G} |S^g|$. (Hint: consider the set of the pairs (g, s) satisfying $g(s) = s$).

7.b (6) Prove Burnside's formula: $|G| \times$ (number of orbits) $= \sum_{g \in G} |S^g|$.

8 The Galois Correspondence

Suppose α is a zero of a monic irreducible polynomial $f \in \mathbb{Q}[x]$ of degree 9. Then, Cauchy's theorem says that the quotient ring $K = \mathbb{Q}[x]/(f(x))$ is a field extension of $\mathbb{Q}$ of degree 9 isomorphic to $\mathbb{Q}(\alpha)$.

8.a (2) Suppose α is a real number, but none of the other zeros of f are real. Explain why K has no (non-trivial) field automorphisms.

8.b (3) Suppose there is a field M properly between K and $\mathbb{Q}$. What are the possible degrees of $M/\mathbb{Q}$?

8.c (5) Suppose the Galois closure of $K/\mathbb{Q}$ in L and $G(L/\mathbb{Q})$ is S_9. Explain why there is no field properly between K and $\mathbb{Q}$.

Additional problems

9 Representations of A_4

Describe all irreducible representations of A_4.

9.a (2) Find the class equation of A_4.

9.b (3) Using the theorem about the characters of irreducible representations and part a) of the problem describe all irreducible representations of A_4.

9.c (5) Take the three dimensional irreducible representation of A_4 and it tensor products with all linear representations. Are we getting new three-dimensional irreducible representations by doing that? Explain.

10 The Group $Cl\left(\mathbb{Q}\left[(-13)^{1/2}\right]\right)$

Compute the group $Cl\left(\mathbb{Q}\left[(-13)^{1/2}\right]\right)$.

10.a (2) Describe all algebraic integers of $\mathbb{Q}\left[(-13)^{1/2}\right]$.

10.b (2) Compute $\mu\left(\mathbb{Q}\left[(-13)^{1/2}\right]\right)$. (Recall $\mu = (2D^{1/2})/\pi$.)

10.c (3) Find all primes p smaller than the integral part of $\mu\left(\mathbb{Q}\left[(-13)^{1/2}\right]\right)$.

10.d (3) Describe all prime ideals P in $\mathbb{Q}\left[(-13)^{1/2}\right]$ that divide all primes p smaller than the integral part of $\mu\left(\mathbb{Q}\left[(-13)^{1/2}\right]\right)$.