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Real Analysis Qualifying Exam
September 17, 2010

Problem #	Points
1	
2	
3	
4	
5	
6	
Total	

Instructions. *Do all problems if possible. Use only one side of each sheet. Do at most one problem on each page. Write your name on every page. Justify your answers. Where appropriate, state without proof results that you use in your solutions.*

Prob. 1. Consider a measure space $(X, \mathcal{A}, \mu)$ and a sequence of measurable sets E_n, $n \in \mathbb{N}$, such that $\sum_n \mu(E_n) < \infty$. Show that almost every $x \in X$ is an element of at most finitely many E_n's.

Prob. 2. Consider a measure space $(X, \mathcal{A}, \mu)$ with $\mu(X) < \infty$, and a sequence $f_n : X \to \mathbb{R}$ of measurable functions such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in X$. Show that for every $\epsilon > 0$ there exists a set E of measure $\mu(E) \leq \epsilon$ such that (f_n) converges uniformly to f outside the set E.

Prob. 3. Suppose that $f \in L^p([0, 1])$ for some $p > 2$. Prove that $g(x) = f(x^2) \in L^1([0, 1])$.

Prob. 4. Assume that $E \subset [0, 1]$ is measurable and for any $(a, b) \subset [0, 1]$,

$$\mu\big(E \cap [a, b]\big) \geq \frac{1}{2}(b - a).$$

Show that $\mu(E) = 1$.

Prob. 5. Let f be a real-valued uniformly continuous function on $[0, \infty)$. Show that if f is Lebesgue integrable on $[0, \infty)$, then $\lim_{x\to\infty} f(x) = 0$.

Prob. 6. Consider the Lebesgue measure space $(\mathbb{R}, \mathfrak{M}_L, \mu_L)$ on $\mathbb{R}$. Let f be an extended real-valued $\mathfrak{M}_L$-measurable function on $\mathbb{R}$. For $x \in \mathbb{R}$ and $r > 0$ let $B_r(x) = \big\{y \in \mathbb{R} : |y - x| < r\big\}$.
With $r > 0$ fixed, define a function g on $\mathbb{R}$ by setting

$$g(x) = \int_{B_r(x)} f(y)\, \mu_L(dy) \quad \text{for } x \in \mathbb{R}.$$

(a) Suppose f is locally μ_L-integrable on $\mathbb{R}$, that is, f is μ_L-integrable on every bounded $\mathfrak{M}_L$-measurable set in $\mathbb{R}$. Show that g is a real-valued continuous function on $\mathbb{R}$.
(b) Show that if f is μ_L-integrable on $\mathbb{R}$ then g is uniformly continuous on $\mathbb{R}$.