Real Analysis Qualifying Exam 2015 Fall: *complete all problems if possible. Use only one side of each sheet. Do at most one problem on each page. Write your name on every page. Justify your answers. Where appropriate, state without proof results that you use in your solutions.*

Problem 1: Let E be a measurable subset of $[0, 2\pi]$. Assume that $f \in C(\mathbb{R})$ is 1-periodic, i.e. $f(x+1) = f(x)$. Compute

$$\lim_{n\to+\infty} \int_E f(nx)\, dx.$$

Justify your answer.

Problem 2: Suppose $f \in L^1[0,1]$ and assume that there exists $C > 0$ such that for all measurable subsets $E \subset [0,1]$, we have

$$\int_E |f(x)|dx \leq C\mu(E)^{\frac{1}{2}}.$$

Show that $f \in L^p[0,1]$ for $1 \leq p < 2$. Show that the statement fails for $p = 2$ by giving a counterexample.

Problem 3: Show that a function $f : \mathbb{R}^n \to \mathbb{R}^+$ is measurable if and only if $\{(x,y); 0 \leq y \leq f(x)\}$ is a measurable subset of $\mathbb{R}^{n+1}$.

Problem 4: Let $f \in L^1(\mathbb{R})$ and set

$$f_h(x) = \frac{1}{2h}\int_{x-h}^{x+h} f(t)dt, \quad h > 0.$$

Show that $f_h \in L^1(\mathbb{R})$ and $f_h \to f$ in $L^1(\mathbb{R})$ as $h \to 0$.

Problem 5: Let $(X, \mathcal{A}, \mu)$ be a measure space and let $f_k : X \to \mathbb{R}$ be a sequence of measurable functions satisfying the following:

$$\int_X |f_k|^2 d\mu \leq 2015, \quad \text{for all } k,$$

and

$$\int_X f_j f_k d\mu = 0, \quad \text{for all } j \neq k.$$

Prove that for all $\beta > 3/2$,

$$\lim_{n\to\infty} \frac{1}{n^\beta} \sum_{k=1}^{n^2} f_k(x) = 0, \quad \text{for a.a. } x \in X.$$

Problem 6: Let $A, B \subset \mathbb{R}^n$ be Lebesgue measurable sets and assume that for every $x \in \mathbb{Q}^n$, there exists a null set N_x such that

$$A + x \subset B \cup N_x.$$

Here $A + x = \{a + x;\, a \in A\}$. Show that if A is not a null set, then the complement of B in $\mathbb{R}^n$ is a null set.