Print Your Name: ______________ ______________
last first

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Qualifying Examination, June 15, 2016
1:00 pm — 3:30 pm, Room PSCB 120

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Notation:

$\mathbf{C}$ denotes the complex plane; $i = \sqrt{-1}$;
$D(z_0, r)$ denotes the open disc in $\mathbf{C}$ centered at z_0 and radius r;
$U = \{z = x + iy : y > 0\}$ is the upper half plane in $\mathbf{C}$.

1. Show that

$$\sum_{n=1}^{\infty} \frac{1}{z^2 + n^2}$$

defines a meromorphic function on $\mathbf{C}$.

2. Show that for a positive integer $n \geq 1$

$$\int_0^\infty \frac{1}{x^{2n}+1}\,\mathrm{dx} = \frac{\pi}{2\mathrm{n}\sin\frac{\pi}{2\mathrm{n}}}.$$

3. For any non-integers α, β and γ, find the radius of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)\beta(\beta+1)\ldots(\beta+n-1)}{n!\gamma(\gamma+1)\ldots(\gamma+n-1)} z^n.$$

4. Let f be an entire function. Prove the following two statements.
(a) If $|f(z)| \leq M(1+|z|^n)$ on $\mathbf{C}$ for some positive constant M then f is a polynomial of degree at most n.

(b) If $\lim_{|z|\to\infty} |f(z)| = \infty$ then f is a polynomial.

5. Find all entire holomorphic functions f with justification such that

$$\text{Im} f(z) = (y^2 - x^2),$$

where $\text{Im} f$ denotes the imaginary part of f.

6. Prove or disprove: there exists a family $\{f_n\}$ of holomorphic functions on $D(0,2)$ such that $f_n \to \bar{z}^3$ uniformly on the compact set $\{z \in \mathbf{C} : |z| = 1 \text{ or } 1/2\}$ (two circles: $|z| = 1$ and $|z| = 1/2$).

7. Construct a conformal map ϕ which maps D_1 onto D_2, where

$$D_1 = \{z = x + iy \in D(0,1) : y > x\}; \text{ and } D_2 = \{z \in \mathbf{C} : |z| > 1\}.$$

8. Let $f : U \to U$ be holomorphic with U being the upper half plane. Prove that

$$|f'(i)| \leq |f(i)|$$

and provide an example indicates the above inequality is an equality.