Print Your Name: ______________ ______________
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Comprehensive Examination of Analysis

9:00Am–11:30AM, June 17, 2014

Table of your scores

Problem 1 ________/ 10

Problem 2 ________/ 10

Problem 3 ________/ 10

Problem 4 ________/ 10

Problem 5 ________/ 10

Problem 6 ________/ 10

Problem 7 ________/ 10

Problem 8 ________/ 10

Problem 9 ________/ 10

Total ________/ 90

Score:———/10 Your Name: ______________ ______________
last first

1. Suppose that f is differentiable on $(-\infty, \infty)$ and has n many zeros in $(-\infty, \infty)$. Prove that $f'(x)$ has $(n-1)$ many zeros in $(-\infty, \infty)$.

Score:———/10 Your Name: __________ __________
last first

2. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points in $\mathbb{R}^m$ such that

$$\sum_{n=1}^{\infty} \|x_n - x_{n-1}\| < \infty.$$

Prove that $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence in $\mathbb{R}^m$.

Score:———/10 Your Name: ____________ ____________
last first

3. Show that the sequence $\{a_n\}_{n=1}^{\infty}$ defined recursively by

$$a_1 > 1, \qquad a_n = \sqrt{2a_{n-1} - 1}, \qquad n \geq 2,$$

converges and finds its limit.

Score:——/10 Your Name: ________ ________
last first

4. (a) Let $F : (0, \infty) \to \mathbb{R}$ be an increasing function which is bounded from above. Prove $\lim_{x\to\infty} F(x)$ exists.

(b) Let f and g two continuous functions on $(0, \infty)$ such that

$$0 \le f(x) \le g(x)$$

If $\int_0^\infty g(x)dx$ exists in $\mathbb{R}$, then $\int_0^\infty f(x)dx$ exists in $\mathbb{R}$.

Score:———/10 Your Name: ________ ________
last first

5. Let S be a subset of $\mathbb{R}^2$ such that every point $x \in S$ is an isolated point. Prove that S is at most countable.

Score:——/10 Your Name: __________ __________
last first

6. Let $X = C[0, 2\pi]$ be the space of all real-valued continuous functions on $[0, 2\pi]$ with a metric

$$d(f, g) = \max\{|f(x) - g(x)| : x \in [0, 2\pi]\}, \quad f, g \in C[0, 2\pi].$$

Let

$$Y = \{\sin(x + \alpha) : \alpha \in \mathbb{R}\} \subset C[0, 2\pi].$$

Prove that Y is a compact subset of (X, d).

Score:———/10 Your Name: ____________ ____________
last first

7. Let $f(x)$ be a Riemann integrable function on $[0,1]$. Prove that

$$\lim_{m\to\infty}\int_0^1 f(x)\cos(mx)dx = 0.$$

Score:———/10   Your Name: ____________ ____________
last first

8. Let

$$f(x) = \ln(1 + \|x\|^2), \quad x \in \mathbb{R}^n.$$

Prove that $f(x)$ is uniformly continuous on $\mathbb{R}^n$.

Score:———/10 Your Name: ____________ ____________
last first

9. Let f be a differentiable function on $[0, 1]$ such that

$$\int_0^1 |f'(s)|^2 ds \leq A^2$$

for some positive constant A. Prove

$$|f(x) - f(y)| \leq A|x - y|^{1/2}$$

for all $x, y \in [0, 1]$.