ALGEBRA COMPREHENSIVE EXAM

Monday, 19 June 2017

Math Exam ID#: ____________________

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total		100

Instructions: Justify your answers. Clearly indicate your final answers, and cross out work that you do not wish to be considered. Each of the 10 questions is worth 10 points. Do as many problems as you can, as completely as you can. The exam is two and one-half hours. No notes, books, or calculators.

1. Does there exist an integer $n \geq 4$ such that the symmetric group S_n is isomorphic to a dihedral group? Prove your answer.
2. a. How many elements x in $\mathbb{Z}/808\mathbb{Z}$ satisfy $x^2 = 1$? Prove your answer.

   b. If k is a finite field and $a, b \in k$, prove that the polynomials $x^2 - a$, $x^2 - b$, $x^2 - ab$ cannot all be irreducible.
3. a. Define Euclidean domain.

   b. Prove that every Euclidean domain is a PID.
4. Let A and B be complex $n \times n$ matrices. Show that $\det(I_n - AB) = \det(I_n - BA)$.
5. Let p, q denote prime integers and let e, f denote positive integers. Assume a finite field of cardinality q^f contains a subring which is a field of cardinality p^e. Prove that $p = q$ and that e divides f.
6. Prove that for every prime number p, the polynomial $x^{p-1} + x^{p-2} + \cdots + x + 1 \in \mathbb{Q}[x]$ is irreducible.
7. Prove that $\mathbb{Q}$ is not a free $\mathbb{Z}$-module.
8. Let V denote a vector space and let W_1, W_2 denote subspaces of V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.
9. Let $T : V \longrightarrow V$ be a $\mathbb{C}$-linear endomorphism of a finite-dimensional complex vector space V. Assume that T has finite order, that is, $T^n = 1$ for some positive integer n. Show that V has a basis consisting of the eigenvectors of T.
10. Answer True or False to each of the following statements; no explanation is necessary.

    a. True/False. If an abelian group is generated by finitely many elements, each of which has infinite order, then the only element of finite order in the group is the identity element.

    b. True/False. Let R denote a commutative ring with unity $1 \neq 0$. There exists a field k and a surjective ring homomorphism $f : R \to k$.

    c. True/False. If p is prime and G is a group of order p^2, then G is abelian.